Imaginary Numbers

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CONNEXIONS

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Chapter 1

Imaginary Concepts – Introduction[1]

$(-1)^2 = 1$

$1^2 = 1$

Whether you square a positive or a negative number, the answer is positive. It is impossible to square **any** number and get a negative answer.

So what is $\sqrt{-1}$? Since it asks the question "What number **squared** is –1?", and since nothing squared ever gives the answer –1, we say that the question has no answer. More generally, we say that the domain of $\sqrt{x}$ is all numbers x such that $x \geq 0$. –1 is not in the domain.

However, it turns out that for a certain class of problems, it is useful to define a new kind of number that has the peculiar property that when you square them, you **do** get negative answers.

Definition of i

The definition of the imaginary number i is that it is the square root of –1:

$i = \sqrt{-1}$ or, equivalently, $i^2 + -1$

i is referred to as an "imaginary number" because it cannot represent real quantities such as "the number of rocks" or "the length of a stick." However, surprisingly, imaginary numbers can be useful in solving many real world problems!

I often like to think of x as being like a science fiction story. Many science fiction stories are created by starting with **one false premise,** such as "time travel is possible" or "there are men on Mars," and then following that premise logically to see where it would lead. With imaginary numbers, we start with the premise that "a number exists whose square is –1." We then follow that premise logically, using all the established rules of math, to see where it leads.

> ***"The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being."***
>
> ***-Gottfried Wilhelm Leibniz***

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[1] This content is available online at <http://cnx.org/content/m18285/1.3/>.

Chapter 2

Imaginary Concepts – Playing with i[1]

Let's begin with a few very simple exercises designed to show how we apply the normal rules of algebra to this new, abnormal number.

A few very simple examples of expressions involving i	

continued on next page

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[1] This content is available online at <http://cnx.org/content/m18286/1.3/>.

Simplify:	$i \bullet 5$
Answer:	5i
Simplify:	$i + 5\mathrm{i}$
Answer:	6i (Add anything to 5 of itself, and you get 6 of it. Or, you can think of this as "pulling out" an i as follows: $i + 5\mathrm{i} = i\,(1 + 5) = 6\mathrm{i}$)
Simplify:	$2\mathrm{i} + 3$
Answer:	You can't simplify it.

Table 2.1

Now let's try something a little more involved.

Example: Simplify the expression (3+2i)2	
$(3 + 2\mathrm{i})^2 = 3^2 + 2\,(3)\,(2\mathrm{i}) + (2\mathrm{i})^2$	**because**$(x + a)\,2 = x^2 + 2\mathrm{ax} + a^2$**as always**
$= 9 + 12\mathrm{i} - -4$	$(2\mathrm{i})^2 = (2\mathrm{i})(2\mathrm{i}) = (2)(2)(\mathrm{i})(\mathrm{i}) = 4\mathrm{i}^2 = - - 4$
$= 5 + 12\mathrm{i}$	**we can combine the** 9 **and** –4, **but not the** 12i.

Table 2.2

It is vital to remember that i is not a variable, and this is not an algebraic generalization. You cannot plug $i = 3$ into that equation and expect anything valid to come out. The equation $(3{+}2\mathrm{i})^2 = 5 + 12\mathrm{i}$ has been shown to be true for only one number: that number is i, the square root of –1.

In the next example, we simplify a radical using **exactly the same technique that we used in the unit on radicals**, except that $a - -1$ is thrown into the picture.

Example: Simplify $\sqrt{-20}$	
$\sqrt{-20} = \sqrt{(4)\,(5)\,(-1)}$	**as always, factor out the perfect squares**
$= \sqrt{4}\sqrt{5}\sqrt{-1}$	**then split it, because**$\sqrt{\mathrm{ab}}$= $\sqrt{a}\sqrt{b}$
$= 2\mathrm{i}\ \sqrt{5}$	$\sqrt{4}$=2, $\sqrt{-1}$= i, **and**$\sqrt{5}$**is just**$\sqrt{5}$
Check	
Is 2i $\sqrt{5}$ really the square root of –20? If it is, then when we **square** it, we should get –20.	
$\left(2\mathrm{i}\sqrt{5}\right)^2 = 2^2 i^2 5^2 = 4 * -1 * 5 = \ -20$ ☑ It works!	

Table 2.3

The problem above has a very important consequence. We began by saying "You can't take the square root of any negative number." Then we defined i as the square root of –1. But we see that, using i, we can now take the square root of **any** negative number.

Chapter 3

Imaginary Concepts – Complex Numbers[1]

A "complex number" is the sum of two parts: a real number by itself, and a real number multiplied by i. It can therefore be written as $a + \mathrm{bi}$, where a and b are real numbers.

The first part, a, is referred to as the **real part**. The second part, bi, is referred to as the **imaginary part**.

Examples of complex numbers $a + \mathrm{bi}$ (a is the "real part"; bi is the "imaginary part")	
$3 + 2\mathrm{i}$	$a = 3$, $b = 2$
π	$a = \pi$, $b = 0$(no imaginary part: a "pure real number")
$-i$	$a = 0$, $b = -1$ (no real part: a "pure imaginary number")

Table 3.1

Some numbers are not obviously in the form $a + \mathrm{bi}$. However, any number can be **put** in this form.

Example 1: Putting a fraction into $a + \mathrm{bi}$ form (i in the numerator)
$\frac{3-4i}{5}$ is a valid complex number. But it is not in the form $a + \mathrm{bi}$, and we cannot immediately see what the real and imaginary parts are.
To see the parts, we rewrite it like this:
$\frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i$
Why does that work? It's just the ordinary rules of fractions, applied backward. (Try multiplying and then subtracting on the right to confirm this.) But now we have a form we can use:
continued on next page

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[1] This content is available online at <http://cnx.org/content/m18282/1.3/>.

$\frac{3-4i}{5}$ $a = \frac{3}{5}$, $b = - - \frac{4}{5}$
So we see that fractions are very easy to break up, if the i is in the numerator. An i in the denominator is a bit trickier to deal with.

Table 3.2

Example 2: Putting a fraction into $a+\mathrm{bi}$ form (i in the denominator)	
$\frac{1}{i} = \frac{1 \cdot i}{i \cdot i}$	**Multiplying the top and bottom of a fraction by the same number never changes the value of the fraction: it just rewrites it in a different form.**
$= \frac{i}{-1}$	**Because $i \bullet i$ is i^2, or –1.**
$= -\mathrm{i}$	**This is not a property of i, but of –1. Similarly,** $\frac{5}{-1} = - - 5$.
$\frac{1}{i}$: $a = 0$, $b = -1$	**since we rewrote it as** $-\mathrm{i}$, **or** $0 - 1\mathrm{i}$

Table 3.3

Finally, what if the denominator is a more complicated complex number? The trick in this case is similar to the trick we used for rationalizing the denominator: we multiply by a quantity known as the **complex conjugate of the denominator**.

Definition of Complex Conjugate

The complex conjugate of the number $a + \mathrm{bi}$ is $a - \mathrm{bi}$. In words, you leave the real part alone, and change the sign of the imaginary part.

Here is how we can use the "complex conjugate" to simplify a fraction.

Example: Using the Complex Conjugate to put a fraction into $a+\mathrm{bi}$ form	
$\frac{5}{3-4i}$	**The fraction: a complex number not currently in the form** $a + bi$
$= \frac{5(3+4i)}{(3-4i)(3+4i)}$	**Multiply the top and bottom by the complex conjugate of the denominator**
$= \frac{15+20i}{3^2-(4i)}{}^2$	**Remember,** $(x+a)(x - -a) = x^2 - - \mathrm{a}^2$
$= \frac{15+20i}{9+16}$	$(4\mathrm{i})^2 = 4^2 i^2 = 16(- - 1) = - - 16$, **which we are subtracting from 9**
$= \frac{15+20i}{25}$	**Success! The top has i, but the bottom doesn't. This is easy to deal with.**
$= \frac{15}{25} + \frac{20i}{25}$	**Break the fraction up, just as we did in a previous example.**
$= \frac{3}{5} + \frac{4}{5}i$	**So we're there!** $a = \frac{3}{5}$ and $b = \frac{4}{5}$

Table 3.4

Any number of any kind can be written as $a + \mathrm{bi}$. The above examples show how to rewrite fractions in this form. In the text, you go through a worksheet designed to rewrite $\sqrt[3]{-1}$ as three different complex numbers. Once you understand this exercise, you can rewrite other radicals, such as $\sqrt{i}$, in $a + \mathrm{bi}$ form.

Chapter 4

Imaginary Concepts – Equality and Inequality in Complex Numbers[1]

What does it mean for two complex numbers to be **equal?** As always, equality asserts that two things are exactly the same. $7 + 3\mathrm{i}$ is **not** equal to 7, or to 3i, or to $7 - 3\mathrm{i}$, or to $3 + 7\mathrm{i}$. It is not equal to anything except $7 + 3\mathrm{i}$.

Definition of Equality

Two complex numbers are **equal** to each other only if **their real parts are equal, and their imaginary parts are equal.**

So if we say that two complex numbers equal each other, we are actually making two separate, independent statements. We can use this, for instance, to solve for two separate variables.

Example: Complex Equality	
If $3\mathrm{x} + 4\mathrm{yi} + 7 = 4\mathrm{x} + 8\mathrm{i}$, what are x and y?	
Normally, it is impossible to solve one equation for two unknowns. But this is really two separate equations!	
Real part on the left = real part on the right:	$3\mathrm{x} + 7 = 4\mathrm{x}$
Imaginary part on the left = imaginary part on the right:	$4\mathrm{y} = 8$
We can now solve both of these equations trivially.	$x = 7$, $y = 2$

Table 4.1

And what about **inequalities?** The answer may surprise you: **there are no inequalities with complex numbers,** at least not in the form we're seeing.

The real numbers have the property that for any two real numbers a and b, exactly one of the following three statements must be true: $a = b$, $a > b$, or $a < b$. This is one of those properties that seems almost too obvious to bother with. But it becomes more interesting when you realize that the complex numbers do **not** have that property. Consider two simple numbers, i and 1. Which of the following is true?

- $i = 1$ ☒
- $i > 1$ ☒
- $i < 1$ ☒

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[1] This content is available online at <http://cnx.org/content/m18283/1.1/>.

None of them is true. It is not generally possible to describe two complex numbers as being “greater than” or “less than” each other.

Visually, this corresponds to the fact that all the real numbers can be laid out on a number line: “greater than” means “to the right of” and so on. The complex numbers cannot be laid out on a number line. They are sometimes pictured on a 2-dimensional graph, where the real part is the x coordinate and the imaginary part is the y coordinate. But one point on a graph is neither greater than, nor less than, another point!

Chapter 5

Imaginary Concepts – Quadratic Equations and Complex Numbers[1]

In the unit on quadratic equations and complex numbers, we saw that a quadratic equation can have **two answers, one answer, or no answers**.

We can now modify this third case. In cases where we described "no answers" there are actually two answers, but both are complex! This is easy to see if you remember that we found "no answers" when the **discriminant was negative**—that is, when the quadratic formula gave us a negative answer in the square root.

As an example, consider the equation:

$2\mathrm{x}^2 + 3\mathrm{x} + 5 = 0$

The quadratic equation gives us:

$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(5)}}{4} = \frac{-3 \pm \sqrt{-31}}{4}$

This is the point where, in the "old days," we would have given up and declared "no answer." Now we can find two answers—both complex.

$= \frac{-3}{4} \pm \frac{\sqrt{31}\sqrt{-1}}{4} = \frac{-3}{4} \pm \frac{\sqrt{31}}{4} i$

So we have two answers. Note that the two answers are **complex conjugates** of each other—this relationship comes directly from the quadratic formula.

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[1] This content is available online at <http://cnx.org/content/m18288/1.2/>.

Chapter 6

Imaginary Concepts – Extra for Experts[1]

6.1 Illegal Operations

So far, we have seen three different **illegal operations** in math.

1. You cannot take the square root of a negative number. (Hence, the domain of $\sqrt{x}$ is $x \geq 0$.)
2. You cannot divide by zero. (Hence, the domain of $\frac{1}{x}$ is $x \neq 0$.)
3. You cannot take the log of 0 or a negative number. (Hence, the domain of $\log(x)$ is $x > 0$.)

Imaginary numbers give us a way of violating the first restriction. Less obviously, they also give us a way of violating the third restriction: with imaginary numbers, you can take the log of a negative number.

So, how about that second restriction? Do you ever reach a point in math where the teacher admits “OK, we **really can** divide by 0 now”? Can we define a new imaginary number $j = \frac{1}{0}$?

The answer is emphatically no: **you really can't divide by 0**. If you attempt to define an imaginary way around this problem, all of math breaks down. Consider the following simple example:

$5 \bullet 0 = 3 \bullet 0$	**That's true**
$5 = 3$	**Divide both sides by 0**

Table 6.1

You see? Dividing by 0 takes us from true conclusions to false ones.

The astonishing thing about the definition $i = \sqrt{-1}$ is that, although it is imaginary and nonsensical, it is **consistent:** it does not lead to any logical contradictions. You can find many ways to simplify $\frac{1}{i}$ and it will always reduce to $- -$ i in the end. Division by zero can never be consistent in this way, so it is always forbidden.

A great deal of Calculus is concerned with getting around this problem, by dividing by numbers that are **very close** to zero.

6.2 The World of Numbers

When you first learn about numbers, you learn the **counting numbers:**

1,2,3,4...	**counting numbers**

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[1] This content is available online at <http://cnx.org/content/m18284/1.3/>.

Table 6.2

These numbers are perfect for answering questions such as “How many sticks do I have?” “How many days until Christmas?” “How many years old are you?”

For other questions, however, you run into limitations. In measuring temperature, for instance, we find that we need lower numbers than 1. Hence, we arrive at a broader list:

...–4,–3,–2,–1,0,1,2,3,4...	**integers**

Table 6.3

The addition of 0 and the negative numbers gives us a new, broader set. The original idea of numbers is now seen as a special case of this more general idea; the original set is a subset of this one.

Still, if we are measuring lengths of sticks, we will find that often they fall **between** our numbers. Now we have to add fractions, or decimals, to create the set of **rational numbers**. I can no longer list the set, but I can give examples.

$\frac{1}{2}$, –3, $\frac{22}{7}$, 0, 2.718, 0.14141414...	**rational numbers**

Table 6.4

The word “rational” implies a **ratio**, or fraction: the ratio of **two integers**. Hence, we define our new, broader set (rational numbers) in terms of our older, more limited set (integers). Rational numbers can be expressed as either fractions, or as decimals (which either end after a certain number of digits, or repeat the same loop of digits forever).

This set seems to be all-inclusive, but it isn’t: certain numbers cannot be expressed in this form.

$\sqrt{2}$, π	**irrational numbers**

Table 6.5

The square root of any non-perfect square is “irrational” and so is π. They can be **approximated** as fractions, but not expressed exactly. As decimals, they go on forever but do not endlessly repeat the same loop.

If you take the rationals and irrationals together, you get the **real numbers**. The real numbers are all the numbers represented on a number line.

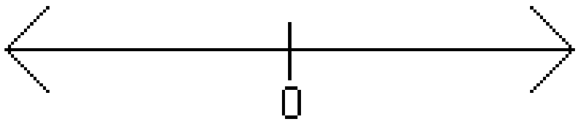

Figure 6.1: All the numbers on a number line are the real numbers

Now, with this unit, we have added the final piece of the puzzle, the **complex numbers**. A complex number is any number $a + \mathrm{b}i$ where a and b are real numbers. Hence, just as our definition of rational numbers was **based on** our definition of integers, so our definition of complex numbers is **based on** our definition of real numbers. And of course, if $b = 0$ then we have a real number: the old set is a subset of the new.

All of this can be represented in the following diagram.

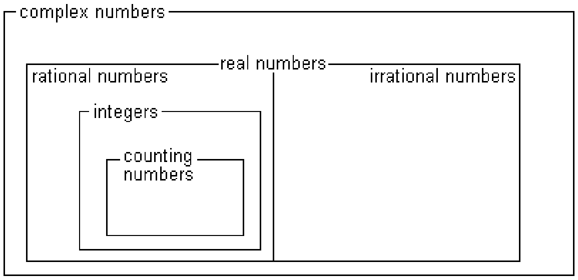

Figure 6.2

The diagram captures the vital idea of subsets: all real numbers are complex numbers, but not all complex numbers are real.

Similarly, the diagram shows that if you take all the rational numbers, and all the irrational numbers, together they make up the set of real numbers.

Index of Keywords and Terms

Keywords are listed by the section with that keyword (page numbers are in parentheses). Keywords do not necessarily appear in the text of the page. They are merely associated with that section. *Ex.* apples, § 1.1 (1) **Terms** are referenced by the page they appear on. *Ex.* apples, 1

A algebra, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)

C complex, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)
complex number, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)

I i, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)
imaginary, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)
imaginary number, § 1(1), § 2(3), § 3(5), § 4(7), § 5(9), § 6(11)

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